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Groups with many subnormal subgroups[☆]

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Abstract

A generalization of groups with all subgroups subnormal is studied. In particular, we prove that a group G with a finite subgroup F such that every subgroup containing F is subnormal of bounded defect, is finite-by-(nilpotent of bounded class) provided it is either locally nilpotent or periodic with $\pi(G)$ finite.

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A subgroup H of a group G is said to be *subnormal* in G if there exists a *subnormal series* from H to G , that is, a finite series

$$H = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$$

of subgroups of G each of those is normal in the following one. The length of the shortest subnormal series from H to G is called the *defect of subnormality* of H in G .

It is easy to see that if G is a nilpotent group of class n , then every subgroup of G is subnormal with defect at most n . A partial converse comes from a fundamental result by Roseblade:

Theorem 0.1 (Roseblade [7, 6.1.3]). *There exists a function $\rho : \mathbb{N} \rightarrow \mathbb{N}$ such that, for every integer n , any group with all subgroups subnormal of defect at most n is nilpotent of class at most $\rho(n)$.*

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On the other hand, celebrated examples by H. Heineken and I.J. Mohamed [6] show that there exist groups with trivial center in which every subgroup is subnormal (with no bound on the defects, clearly).

In this paper we consider groups in which many (but not necessarily all) subgroups are assumed to be subnormal with bounded defect. We prove that if G is a locally nilpotent group with a finite subgroup F such that every subgroup containing F is subnormal of bounded defect, then G is finite-by-(nilpotent of bounded class). More precisely, denoting by $\gamma_n(G)$ the n th term of the lower central series of G , we get:

Theorem 0.2. *There exists a function $\beta(n)$ of n , such that if G is a locally nilpotent group and there exists a finite subgroup F of G with the property that every subgroup of G containing F is subnormal of defect at most n in G , then $\gamma_{\beta(n)}(G)$ is finite. In particular, G is nilpotent and its nilpotency class is bounded by a function depending on n and on the order of F .*

The motivation for such an investigation comes from its possible applications to the theory of groups with all subgroups subnormal. For example, our Theorem 0.2 has in fact been applied by Casolo to prove that such groups are metanilpotent and Fitting groups (see [1,2]). Another application is given in [4].

We also prove a result corresponding to Theorem 0.2 on periodic groups with a finite number of involved primes. More precisely, denoting by $\pi(G)$ the set of all primes dividing the order of the elements of G , we have:

Theorem 0.3. *There exists a function $\beta(n)$ of n , such that if G is a torsion group with $\pi(G)$ finite and there exists a finite subgroup F of G with the property that every subgroup H of G containing F is subnormal of defect at most n in G , then $\gamma_{\beta(n)}(G)$ is finite.*

Note that the condition on the finiteness of $\pi(G)$ cannot be relaxed:

Proposition 0.1. *There exists a torsion group G with a finite subgroup F such that every subgroup H containing F is subnormal of defect at most 2 in G , and G is not finite-by-nilpotent.*

Also, relaxing the condition that F is finite does not lead anymore to finite-by-nilpotent groups, as Smith's groups [10] show. However, we shall prove that if G is a locally nilpotent group with a finitely generated subgroup F such that every subgroup containing F is subnormal of bounded defect, then G is hypercentral (Proposition 2.1).

As a related result, we show that a finitely generated group G , with a finite subgroup F such that every subgroup containing F is subnormal in G (with no prescribed bound on the defects), is finite-by-nilpotent (Proposition 1.1).

1. Preliminary results

For brevity, we write $H \text{ sn}_n G$ to mean that H is a subnormal subgroup of G of defect at most n in G . As the class of groups with every subgroup subnormal of defect at most n is usually denoted by \mathfrak{U}_n , we call \mathfrak{U}_n^+ the class of all groups G in which there exists a finite subgroup F such that $H \text{ sn}_n G$ whenever $F \leq H \leq G$. By abuse of notation, we shall write the above by $(G, F) \in \mathfrak{U}_n^+$.

Note that \mathfrak{U}_n^+ -groups are finite-by-soluble. Indeed, if $(G, F) \in \mathfrak{U}_n^+$, then F is subnormal in G and every section of its subnormal series belongs to \mathfrak{U}_n and so, by Roseblade's Theorem 0.1, every section is nilpotent. This also implies that torsion \mathfrak{U}_n^+ -groups are locally finite.

We begin with a result on finitely generated groups; note that this proposition still holds if it is only assumed that G/G' (or G/F^G) is finitely generated.

Proposition 1.1. *Let G be a finitely generated group with a finite subgroup F such that every subgroup of G containing F is subnormal in G . Then G is finite-by-nilpotent.*

Proof. Since the quotient group of G over $N = F^G$ is nilpotent, it is sufficient to prove that N is finite.

Case 1. $F \leq A \trianglelefteq G$ where A is an elementary abelian p -group, for a prime p , and $G/A = \langle x \rangle$ is cyclic.

The proof proceed by induction on the minimal number d of generators of the finite subgroup F .

If $d = 1$, say $F = \langle y \rangle$, then $N = F^G = \langle y^{x^i} \mid i \in \mathbb{Z} \rangle$. Set $E = \langle y \rangle^{\langle x^2 \rangle}$ and $D = \langle y^x \rangle^{\langle x^2 \rangle}$. Thus, E and D are normal subgroup of $N\langle x^2 \rangle$ and $ED = N$.

If $E \cap D = 1$, then $N/E = DE/E \simeq D$. Since $F \leq E$, in $N\langle x^2 \rangle/E$ every subgroup is subnormal. Moreover, $N\langle x^2 \rangle$ has finite index in G ; hence $N\langle x^2 \rangle/E$ is finitely generated and nilpotent. In particular, $N/E \simeq D$ is finitely generated and so it is finite. Therefore, since the conjugation by x is an isomorphism between E and D , we conclude that N is finite.

Assume now that $E \cap D \neq 1$. Then, some elements $y^{x^{\alpha_1}}, y^{x^{\alpha_2}}, \dots, y^{x^{\alpha_r}}$, for $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_r$ integers and $r \in \mathbb{N}$, are linearly dependent on a field with p elements. In particular,

$$y^{x^{\alpha_1}} \in \langle y^{x^i} \mid \alpha_1 < i \leq \alpha_r \rangle = L_1, \quad y^{x^{\alpha_r}} \in \langle y^{x^i} \mid \alpha_1 \leq i < \alpha_r \rangle = L_r.$$

Thus

$$\begin{aligned} y^{x^{\alpha_1-1}} &= (y^{x^{\alpha_1}})^{x^{-1}} \in \langle y^{x^{i-1}} \mid \alpha_1 < i \leq \alpha_r \rangle = \langle y^{x^i} \mid \alpha_1 \leq i < \alpha_r \rangle = L_1, \\ y^{x^{\alpha_r+1}} &= (y^{x^{\alpha_r}})^x \in \langle y^{x^{i+1}} \mid \alpha_1 \leq i < \alpha_r \rangle = \langle y^{x^i} \mid \alpha_1 < i \leq \alpha_r \rangle = L_r, \end{aligned}$$

since $y^{x^{\alpha_1}} \in L_1$ and $y^{x^{\alpha_r}} \in L_r$. It is easy to prove, by induction, that $y^{x^{\alpha_1-j}} \in L_1$ and $y^{x^{\alpha_r+j}} \in L_r$ for every $j \geq 0$, that is, $N \leq L_1 L_r$. This implies that N is finite.

Let now $d > 1$ and $F = \langle y, y_2, \dots, y_d \rangle$. Since $F\langle y \rangle^G / \langle y \rangle^G$ is generated by $d - 1$ elements, the induction hypothesis gives that $G\langle y \rangle^G$ is finite-by-nilpotent. Thus, if $\langle y \rangle^G$ is finite, then G is finite-by-nilpotent.

Otherwise, $\langle y \rangle^G$ is infinite and, by the same arguments used before, we get that $E = \langle y \rangle^{\langle x^2 \rangle}$ has trivial intersection with $D = \langle y^x \rangle^{\langle x^2 \rangle}$. Now, since $y \in E$ and $N\langle x^2 \rangle$ has finite index in G , the group $G_1 = N\langle x^2 \rangle / E$ is, by induction, finite-by-nilpotent and finitely generated. In particular, $D \simeq N/E$ is finite and, since $E \simeq D$, $N = ED$ is finite.

Case 2. Assume now that $N = F^G$ is abelian.

This implies that $\pi(N) = \pi(F)$ is finite and that N is a finite direct product of its p -components N_p , for $p \in \pi(N)$. Since G is isomorphic to a subgroup of the direct products of the $G / \prod_{q \neq p} N_q$, for $p \in \pi(N)$, we can clearly assume that N is a p -group, for a prime p .

Let $\bar{x} = xN^p$ be an element of G/N^p and let $\bar{F} = FN^p/N^p$. By point 1, $\langle \bar{x}, \bar{F} \rangle$ is finite-by-nilpotent. Thus, G/N^p is generated by a finite number of subnormal finite-by-nilpotent subgroups and, being finitely generated, it is finite-by-nilpotent by [7, Theorem 1.6.2]. In particular, G/N^p satisfies the maximal condition on subgroups and hence N/N^p is finite. Finally, since N , being abelian, has finite exponent, it is a direct product of cyclic groups (see [9, 4.3.5]) and the condition on N/N^p forces N to be finite.

Case 3. The general case.

By point 2, G/N' is finite-by-nilpotent and satisfies the maximal condition on subgroups. In particular, N/N' is finitely generated. Let τ be a left transversal to N' in N and set $X = \langle \tau, F \rangle$. So, X is finitely generated and subnormal in N . If $N \neq X$, then there exists a normal proper subgroup M of N containing X . Now, since every subgroup of N/M is subnormal, N/M is soluble, by a well-known result by Möhres. Hence, there exists a normal subgroup L of N containing X such that N/L is abelian. Therefore $N' \leq L < N = N'X$ and this gives a contradiction.

Thus, N is finitely generated and, by induction on the defect of subnormality of F , we get that N is finite-by-nilpotent. In particular, N is periodic, hence finite, and the proof is complete. \square

We use standard notation for commutators. Thus, if G is a group and $x, y \in G$, then $[x, {}_0 y] = x$, $[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$, and $[x, {}_k y] = [[x, {}_{k-1} y], y]$ for all positive integers k . If A is a subgroup of G and $x \in G$, we set $[A, x] = \langle [a, x] \mid a \in A \rangle$.

We recall some known results.

Lemma 1.1. *Let G be a group and let A be a normal abelian subgroup of G . Let $a, b \in A$, $x, y \in G$ and $n, m \in \mathbb{N}$. Then:*

- (a) $[a, xy] = [a, x][a, y][a, x, y]$ and, if $G/C_G(A)$ is abelian, then $[a, x, y] = [a, y, x]$;
- (b) $[ab, x] = [a, x][b, x]$; hence $[a^n, x] = [a, x]^n$, for all $n \geq 0$, and $[A, x] = \{[a, x] \mid a \in A\}$;

- (c) $[a, x^n] = \prod_{i=1}^n [a, {}_i x]^{(n)}_i$;
 (d) $[A, {}_m x^n] \leq [A, {}_m x]$; in particular, $[A, {}_m x] = 1$ implies $[A, {}_m x^n] = 1$.

It is also well known that if A is a normal elementary abelian p -subgroup of a group G with p a prime, then $[A, {}_{p^n} x] = [A, x^{p^n}]$ for every $x \in G$. If A has a finite exponent p^r for some integer r , one can generalize this fact as in the following lemma.

Lemma 1.2. *Let G be a group and A be a normal abelian subgroup of G with finite exponent p^r . Let $x \in G$.*

- (1) *If $[A, x] \leq A^p$, then $[A, {}_r x] = 1$.*
 (2) *If $[A, {}_{p^n} x] = 1$, then $[A, x^{p^{nr}}] = 1$.*
 (3) *If $[A, x] \leq A^p$, then $[A, x^{p^{r[\log_p r + 1]}}] = 1$.*

Proposition 1.2. *Let G be a group and A be a normal abelian subgroup of G such that $G/C_G(A)$ is abelian. There exists a function $\psi(n, m)$, for n, m positive integers, such that if $|[A, x_1, x_2, \dots, x_n]| \leq m$ for all $x_1, x_2, \dots, x_n \in G$, then $|[A, {}_{2n} G]| \leq \psi(n, m)$.*

Proof. We set $\psi(1, m) = (m!)^2$ for all $m \geq 1$, and for $n > 1$, we define recursively $\psi(n, m) = (\psi(n-1, m)!)^2$. We shall prove that $|[A, {}_{2n} G]| < \psi(n, m)$ by induction on n .

Note that $[A, x]$ is normal in G for every $x \in G$, since $G/C_G(A)$ is abelian.

Let $n = 1$. Clearly, if $[A, x, y] = 1$ for all $x, y \in G$, then $[A, {}_2 G] = 1$. Otherwise, there exist two elements $x, y \in G$ such that $[A, x, y] \neq 1$. Set $N = [A, x][A, y]$, $\bar{G} = G/N$, $\bar{A} = A/N$, and $\bar{x} = xN$.

Suppose that there exists an element $\bar{z} = zN \in \bar{G}$ such that $|[\bar{A}, \bar{z}]| = m$. Since $|[\bar{A}, \bar{z}]| = |[A, z]/[A, z] \cap N|$ and, by assumption, $|[A, z]| \leq m$, we get that $[A, z] \cap N = 1$. In particular, $[N, z] \leq N \cap [A, z] = 1$, which implies that

$$[A, x, z] = [A, y, z] = 1.$$

Also, $[A, x] \leq N$ gives that $|[\bar{A}, \bar{x}\bar{z}]| = |[\bar{A}, \bar{x}][\bar{A}, \bar{z}]| = |[\bar{A}, \bar{x}, \bar{z}]| = |[\bar{A}, \bar{z}]| = m$ and so, by the same argument used before, it follows that $N \cap [A, xz] = 1$. In particular, $[A, y, xz] = 1$. Hence, if a is an element of A such that $[a, x, y] \neq 1$, then, by Lemma 1.1,

$$1 = [a, y, xz] = [a, xz, y] = [[a, x][a, z][a, x, z], y] = [a, x, y][a, z, y] = [a, x, y] \neq 1,$$

which is a contradiction.

Therefore we have $|[\bar{A}, \bar{z}]| \leq m-1$ for every $\bar{z} \in \bar{G}$. By induction on m , we get that $|[\bar{A}, {}_2 \bar{G}]| \leq \psi(1, m-1) = ((m-1)!)^2$ and so

$$|[A, {}_2 G]| \leq |[\bar{A}, {}_2 \bar{G}]||N| \leq ((m-1)!)^2 m^2 = \psi(1, m).$$

Let now $n > 1$. If we fix an element x in G , then our assumption gives that

$$|[[A, x], x_2, x_3, \dots, x_n]| \leq m$$

for every $x_2, x_3, \dots, x_n \in G$. Since $[A, x]$ and $G/C_G([A, x])$ are abelian, the induction hypothesis gives that

$$|[[A, x]_{2(n-1)} G]| \leq \psi(n-1, m).$$

But $[[A, x]_{2(n-1)} G] = [[A, 2(n-1) G], x]$, by Lemma 1.1, and so by case $n = 1$ we conclude that

$$|[A, 2n G]| = |[[A, 2(n-1) G]_2 G]| \leq \psi(1, \psi(n-1, m)) = \psi(n, m),$$

which is our claim. \square

A slight change in the proof of Lemma 1 in [3] gives the following result.

Lemma 1.3 (Casolo [3]). *Let $G = A\langle x \rangle$ be a nilpotent group and $A \trianglelefteq G$ be an elementary abelian p -group. Assume that there exists a finite subgroup F of A such that $(G, F) \in \mathfrak{U}_n^+$, and let $|F| = p^k$. Then $[A, f_p(k, n) - 1 x] = 1$, where $f_p(k, n) = (n+2)p^{\lceil \log_p k(n+2) \rceil}$.*

Note that a finite-by-nilpotent group, generated by a finite number of periodic elements, is periodic. It follows that if G is a group in which every finitely generated subgroup is finite-by-nilpotent, then the elements of finite order in G form a normal subgroup T (the torsion subgroup of G) such that G/T is a locally nilpotent torsion-free group.

To shorten our notation, we denote by $\mathfrak{F}\mathfrak{N}_m$ the class of groups G with a normal subgroup N such that N is finite and G/N is nilpotent of class at most m , that is, such that $\gamma_{m+1}(G)$ is finite.

Lemma 1.4. *Let $G \in \mathfrak{U}_n^+$. If there exists a subgroup A with finite index in G such that $A \in \mathfrak{F}\mathfrak{N}_m$ then $G \in \mathfrak{F}\mathfrak{N}_{nm}$.*

Proof. Since the core $A_G = \bigcap_{g \in G} A^g$ of A has finite index in G and $\gamma_{m+1}(A_G) \leq \gamma_{m+1}(A)$ is a finite normal subgroup of G , we can assume, without loss of generality, that A is normal and nilpotent of class at most m .

Let F be a finite subgroup of G such that $(G, F) \in \mathfrak{U}_n^+$ and let τ be a left transversal to A in G .

Now, set $H = \langle \tau, F \rangle$. Observe that H is finite-by-nilpotent, by Proposition 1.1, and subnormal of defect at most n in G . Let $d = d(G : H) \leq n$ be the defect of subnormality of H in G . We prove by induction on d that $\gamma_{dm+1}(G)$ is finite.

If $d = 1$, then $G/H = AH/H \cong A/A \cap H$ has nilpotency class at most m . Let T be the torsion subgroup of G . Since G/T is a torsion-free locally nilpotent group and A has finite index in G , by [8, Lemma 6.33] G/T is nilpotent of class at most m . Thus, $G/(H \cap T)$ has nilpotency class at most m and $\gamma_{m+1}(G) \leq H \cap T$ is finite, since it is a torsion subgroup of a finitely generated finite-by-nilpotent group.

Now let $d > 1$. Note that $H^G \in \mathfrak{U}_n^+$ and that $A \cap H^G$ is a normal subgroup with finite index in H^G . Since $d(H^G : H) = d - 1$, the induction hypothesis, applied on H^G , implies

that $|\gamma_{(d-1)m+1}(H^G)| < \infty$. Therefore, by Fitting's theorem applied on $G = AH^G$, we conclude that

$$|\gamma_{dm+1}(G)| \leq |\gamma_{(n-1)m+1}(H^G)| \cdot |\gamma_{m+1}(A)| < \infty,$$

and the proof is done. \square

A well-known theorem of P. Hall [5] states that a group G with a normal subgroup N such that N and G/N' are nilpotent of class c and d , respectively, is nilpotent of class at most $g(d, c) = \binom{c+1}{2}d - \binom{c}{2}$. For \mathfrak{U}_n^+ -groups the following holds.

Lemma 1.5. *Let $G \in \mathfrak{U}_n^+$. There exists a function $f(d, c, n)$ such that if $N \trianglelefteq G$ with $N \in \mathfrak{F}\mathfrak{N}_c$ and $G/N' \in \mathfrak{F}\mathfrak{N}_d$, then $G \in \mathfrak{F}\mathfrak{N}_{f(d, c, n)}$.*

Proof. Since $\gamma_{c+1}(N)$ is finite and normal in G , we can assume, without loss of generality, that N is nilpotent of class at most c . By a theorem of P. Hall [8, Theorem 4.25] applied to G/N' , we have that $A/N' = \zeta_{2d}(G/N')$ has finite index in G . Since A/N' has nilpotency class at most $2d$, and AN/N' is the product of A/N' and N/N' , Fitting's theorem gives that AN/N' is nilpotent of class at most $2d + 1$. Then, by the above mentioned Hall's theorem [5] we get that AN is nilpotent of class at most $m = \binom{c+1}{2}(2d + 1) - \binom{c}{2}$. Finally, as AN has finite index in G , by Lemma 1.4 we conclude that $G \in \mathfrak{F}\mathfrak{N}_{nm}$. \square

2. Proofs

Firstly, we prove that certain metabelian locally nilpotent groups belonging to \mathfrak{U}_n^+ are finite-by-nilpotent.

Proposition 2.1. *Let G be a locally nilpotent group and F a finite subgroup of G such that $H \text{ sn}_n G$ whenever $F \leq H \leq G$. Assume also that there exists an abelian normal subgroup A of G such that $F \leq A$ and G/A is abelian. Then $\gamma_{2\rho(n)+2}(G)$ is finite, where $\rho(n)$ is the Roseblade's function (see Theorem 0.1). Moreover, G is nilpotent and its nilpotency class is bounded by a function of n and $|F|$.*

Proof. Denote by $N = F^G$ the normal closure of F in G . By hypothesis $N \leq A$ and $C_G(N) \geq A$, so that N and $G/C_G(N)$ are abelian groups. This implies that N is periodic and $\pi(N) = \pi(F)$ is finite. Clearly N is the direct product of its primary components, say N_p , for $p \in \pi(F)$. If we set $N_{p'} = \prod_{p \neq q \in \pi(F)} N_q$, for $p \in \pi(F)$, then $N_{p'}$ is normal in G and $\bigcap_{p \in \pi(F)} N_{p'} = 1$. Thus G is isomorphic to a subgroup of the finite direct product of the factor groups $G/N_{p'}$ and it is sufficient to prove that $\gamma_{2\rho(n)+2}(G/N_{p'})$ is finite for every $p \in \pi(F)$. Therefore, we can assume that N is a p -group.

Let $|F| = p^k$ and let p^r be the exponent of F : observe that p^r is also the exponent of N .

Consider the elementary abelian p -section $\bar{N} = N/N^p$ in the quotient group $\bar{G} = G/N^p$. Note that $(\bar{G}, \bar{F}) \in \mathfrak{U}_n^+$, where $\bar{F} = FN^p/N^p$, and $|\bar{F}| \leq p^k$.

Let $x \in G$ and $\bar{x} = xN^p$. By assumption, the finitely generated subgroup $\langle \bar{F}, \bar{x} \rangle$ is nilpotent and subnormal in \bar{G} . So $\langle \bar{N}, \bar{x} \rangle$ is nilpotent, since it is the product of the abelian normal subgroup \bar{N} by the nilpotent subnormal subgroup $\langle \bar{F}, \bar{x} \rangle$. Thus, Lemma 1.3 yields $[\bar{N},_{f_p(k,n)-1} \bar{x}] = 1$. Let $s = s_p(k, n)$ be the smallest power of p greater than $f_p(k, n) - 1$: since \bar{N} is an elementary abelian p -group, we have $[\bar{N},_s \bar{x}] = [\bar{N}, \bar{x}^s] = 1$, so that $[N, x^s] \leq N^p$. Then Lemma 1.2 gives $[N, x^{s p^{r[\log_p r+1]}}] = 1$, where p^r is the exponent of F . Set $m = m_p(n, k, r) = s p^{r[\log_p r+1]}$. Since m does not depend on the choice of $x \in G$, we have that $[N, x^m] = 1$ for all $x \in G$. Thus the exponent of $G/C_G(N)$ is at most m .

Now take $\rho = \rho(n)$ elements of G , say x_1, x_2, \dots, x_ρ , and consider the subgroup

$$H = \langle F, x_1, x_2, \dots, x_\rho \rangle = F^H \langle x_1, x_2, \dots, x_\rho \rangle.$$

Since $F^H \leq N \leq C_G(N)$, the quotient group $H/C_H(N)$ is generated by the ρ images of x_1, x_2, \dots, x_ρ . Moreover, $H/C_H(N)$, being isomorphic to a subgroup of $G/C_G(N)$, is an abelian group of exponent bounded by m . Thus the order of $H/C_H(N)$ is at most m^ρ and, in particular, we get

$$|F^H| \leq |F|^{m^\rho}.$$

Now, since $F \leq F^H \trianglelefteq HA$, our assumption on F implies that the group HA/F^H has every subgroup subnormal of defect at most n and so, by Roseblade's theorem, HA/F^H is nilpotent of class bounded by ρ . Thus $[A,_{\rho} H] \leq F^H$ and in particular

$$|[A, x_1, x_2, \dots, x_\rho]| \leq |F^H| \leq |F|^{m^\rho}.$$

Since this bound is independent on the choice of the x_i 's, by Proposition 1.2, we get

$$|[A,_{2\rho} G]| \leq \psi(\rho, |F|^{m^\rho}).$$

Therefore, since $\gamma_2(G) = G' \leq A$, it follows that

$$|\gamma_{2\rho+2}(G)| = |[\gamma_2(G),_{2\rho} G]| \leq \psi(\rho, |F|^{m^\rho}),$$

which proves that $\gamma_{2\rho(n)+2}(G)$ is finite and its order is bounded by a function of n and $|F|$.

Finally, since a theorem of Mal'cev and McLain [9, 12.1.6] states that each principal factor of a locally nilpotent group is central, we get that

$$\gamma_{2\rho+2}(G) \leq \zeta_{\psi(\rho, |F|^{m^\rho})}(G).$$

Thus G is nilpotent and its nilpotency class is bounded by a function of n and $|F|$, which is the desired conclusion. \square

Proof of Theorem 0.2. Let G be a locally nilpotent group and F a finite subgroup of G with the property that $H \text{ sn}_n G$ whenever $F \leq H \leq G$.

Set $\mu(1) = 2\rho(n) + 1$ and define recursively the function μ by $\mu(l) = f(2\rho(n) + 1, \mu(l-1), n)$, where f is the function defined in Lemma 1.5 and ρ is Roseblade's function. Then, the function β is given by $\beta(1) = 3$ and $\beta(n) = f(\mu(\rho(n)), \beta(n-1) - 1, n) + 1$.

We shall prove, by induction on n , that $\gamma_{\beta(n)}(G)$ is finite.

If $n = 1$, then every subgroup H containing F is normal in G . In particular, F is normal in G and, by Dedekind–Baer theorem [7, 6.1.1], G/F is nilpotent of class at most 2. So, $\gamma_{\beta(1)}(G) \leq F$ is finite.

Let now $n > 1$ and set $N = F^G$. Note that, if H is a subgroup of N and $H \geq F$, then $H^G = N$ and so H has defect of subnormality at most $n-1$ in N . This proves that (N, F) belongs to \mathfrak{U}_{n-1}^+ , and so, by induction, we get

$$|\gamma_{\beta(n-1)}(N)| < \infty.$$

Now we consider G/N' : observe that, by Lemma 1.5, it is sufficient to prove that $G/N' \in \mathfrak{F}\mathfrak{N}_{\mu(\rho(n))}$. Let $\rho = \rho(n)$. By our hypothesis, all subgroups of G/N are subnormal of defect at most n . So, by Roseblade's theorem, G/N is nilpotent of class at most ρ . In particular, the derived length l of G/N is bounded by $\log_2(\rho) \leq \rho$.

Now we prove by induction on $l \leq \rho$ that $G/N' \in \mathfrak{F}\mathfrak{N}_{\mu(l)}$.

If $l = 1$, then G/N' is a metabelian group which satisfies the assumption of Proposition 2.1. Hence $\gamma_{\mu(1)+1}(G/N') = \gamma_{2\rho+2}(G/N')$ is finite.

So, let $l > 1$. As the derived length of $G'N/N$ is $l-1$, the induction hypothesis, applied on $(G'N/N', FN'/N') \in \mathfrak{U}_n^+$, gives that

$$G'N/N' \in \mathfrak{F}\mathfrak{N}_{\mu(l-1)}.$$

Now, since $G/(G'N)'$ is a metabelian group and it satisfies the assumption of Proposition 2.1, $\gamma_{2\rho+2}(G/(G'N)')$ is finite.

Therefore, by Lemma 1.5 applied to the quotient group $G/(G'N)' \in \mathfrak{F}\mathfrak{N}_{2\rho+1}$ and to $G'N/N' \in \mathfrak{F}\mathfrak{N}_{\mu(l-1)}$, since $\mu(l) = f(2\rho+1, \mu(l-1), n)$, we conclude that

$$G/N' \in \mathfrak{F}\mathfrak{N}_{\mu(l)}.$$

Clearly, as $\mu(l) \leq \mu(\rho)$, we get $G/N' \in \mathfrak{F}\mathfrak{N}_{\mu(\rho)}$.

Finally, again by applying Lemma 1.5 to $G/N' \in \mathfrak{F}\mathfrak{N}_{\mu(\rho)}$ and to $N \in \mathfrak{F}\mathfrak{N}_{\beta(n-1)}$, we get

$$G \in \mathfrak{F}\mathfrak{N}_{f(\mu(\rho), \beta(n-1), n)},$$

that is, $|\gamma_{\beta(n)}(G)| < \infty$, which is the desired conclusion.

Moreover, applying Hall's theorem [5] instead of Lemma 1.5 in the above construction, we easily get that G is nilpotent and that its nilpotency class is bounded by a function of n and $|F|$. This completes the proof. \square

The following result is analogous to Proposition 2.1, dealing with torsion groups instead of locally nilpotent groups.

Proposition 2.2. *Let G be a torsion group with $\pi(G)$ finite and let F be a finite subgroup of G such that $H \text{ sn}_n G$ whenever $F \leq H \leq G$. Assume also that there exists an abelian normal subgroup A of G such that $F \leq A$ and G/A is abelian. Then $\gamma_{2\rho(n)+2}(G)$ is finite.*

Proof. Let $N = F^G$ be the normal closure of F in G and let $C = C_G(N)$ be the centralizer of N in G . Clearly, N and G/C are abelian groups and $\pi(N) = \pi(F)$. As in the proof of Proposition 2.1, we easily reduce to the case that N is a p -group.

We first prove that G/C has finite exponent. To this end, consider an element xC of G/C . As G/C is abelian and $\pi(G/C) \subseteq \pi(G)$ is finite, we can assume that xC is a q -element for $q \in \pi(G)$.

If $q = p$, then $\langle F, x \rangle$ is a nilpotent group, because it is finite and, modulo its center, it is a p -group. Also, by the assumption on F , $\langle F, x \rangle$ is subnormal in G . Therefore $N\langle x \rangle$, being the product of the abelian normal subgroup N and $\langle F, x \rangle$, is nilpotent. Now, since N/N^p is an elementary abelian p -group and $(N, F) \in \mathfrak{U}_n^+$, Lemma 1.3 gives $[N/N^p, f_p(k, n)-1 x N^p] = 1$, where k is defined by $|F| = p^k$. If $s = s_p(k, n)$ is the smallest power of p greater than $f_p(k, n) - 1$, then $[N, x^s] \leq N^p$ and so Lemma 1.2 yields that $[N, x^{sp^{r[\log_p r+1]}}] = 1$, where p^r is the exponent of N . Indeed, as N is abelian, p^r is precisely the exponent of F . As $m_p = sp^{r[\log_p r+1]}$ depends only on n and on F , this proves that the m_p -power of every p -element of G centralizes N .

We now turn to the case $q \neq p$. Since we are interested on the exponent of G/C and $\langle x \rangle \cap C$ is central in $N\langle x \rangle$, we can assume that $\langle x \rangle \cap C$ is trivial. Let y be the power of x such that the order of y is precisely q . We claim that $[N, y, y] = [N, y]$. Indeed, as $\langle yC \rangle$ is normal in G/C , the subgroup $[N, y, y]$ is normal in G . Also, by Lemma 1.1, for every $z \in N$ we have $1 = [z, y^q] = [z, y]^q$ modulo $[N, y, y]$, so that $[z, y]^q \in [N, y, y]$. Thus, as N is a p -group and $q \neq p$, we conclude that $[N, y] = [N, y, y]$.

Now, the assumption on F implies that $\langle F, y \rangle$ is subnormal of defect at most n in G . Hence $[N, y] \leq \langle F, y \rangle$ and

$$[N, y] = [N, y] \leq N \cap F^{\langle y \rangle} \langle y \rangle = F^{\langle y \rangle} (N \cap \langle y \rangle) = F^{\langle y \rangle},$$

since $N \cap \langle y \rangle \leq C \cap \langle x \rangle = 1$. Also, as $F^{\langle y \rangle}$ is an abelian group generated by the subgroups F^{y^i} , for $i = 1, \dots, q$, the subgroup $[N, y]$ is finite of order at most $|F|^q = p^{kq}$. Now, since $C_{\langle x \rangle}([N, y]) = C_{\langle x \rangle}(N) = 1$, we get that the order of x is bounded by $m_q = p^{kq}!$.

Therefore, since $\pi(G)$ is finite, the exponent of the quotient group G/C is bounded by $m = \prod_{q \in \pi(G)} m_q$.

Now, as in the proof of Proposition 2.1, we take $\rho = \rho(n)$ elements of G , say x_1, x_2, \dots, x_ρ , and we consider the subgroup

$$H = \langle F, x_1, x_2, \dots, x_\rho \rangle = F^H \langle x_1, x_2, \dots, x_\rho \rangle.$$

Since $F^H \leq N \leq C$ and $H/H \cap C$ is isomorphic to a subgroup of the abelian group G/C , we get that $H/C_H(N)$ is an abelian ρ -generated group of exponent at most m . Thus $|H/C_H(N)| \leq m^\rho$ and $|F^H| \leq |F|^{m^\rho}$. Now, by Roseblade's theorem, H/F^H is nilpotent

of class at most n . In particular, $|[A, x_1, x_2, \dots, x_\rho]| \leq |F^H| \leq |F|^{m^\rho}$. By Proposition 1.2, it follows that $|[A, {}_{2\rho}G]| < \infty$, and as $\gamma_2(G) \leq A$, we get

$$|\gamma_{2\rho+2}(G)| = |[\gamma_2(G), {}_{2\rho}G]| < \infty,$$

which is the desired conclusion. \square

Proof of Theorem 0.3. This follows the same lines of the proof of Theorem 0.2, applying Proposition 2.2 instead of Proposition 2.1. \square

Proof of Proposition 0.1. Let p_1, p_2, \dots be an infinite sequence of distinct odd primes and let C_{p_i} be the cyclic group of order p_i . For every $i \in \mathbb{N}$, let M_i be an elementary abelian 2-group on which C_{p_i} acts irreducibly and nontrivially. Note that if $1 \neq y_i \in M_i$, then $[y_i, C_{p_i}] = M_i$. Let

$K = \text{Dr}_{i \in \mathbb{N}} C_{p_i}$, the direct product of the C_{p_i} 's;

$A = \text{Cr}_{i \in \mathbb{N}} M_i$, the Cartesian product of the M_i 's;

$G = K \ltimes A$, the semidirect product of A by K where each C_{p_i} acts as above on M_i and trivially on M_j if $j \neq i$.

Note that, for every element $y \in A$ and $x \in K$, the commutator $[x, y]$ belongs to $\text{Dr}_{i \in \mathbb{N}} M_i$. In particular, $[A, K] = \text{Dr}_{i \in \mathbb{N}} M_i$.

Let us fix an element $y = (y_1, y_2, \dots)$ of A , with $y_i \neq 1$ for all $i \in \mathbb{N}$, and let us set $F = \langle y \rangle$. Clearly $|F| = 2$.

Observe that, if X is a subgroup of K , then $X = \text{Dr}_{i \in I} C_{p_i}$ for a suitable subset $I \subseteq \mathbb{N}$ and $[A, X] = \text{Dr}_{i \in I} M_i$. Moreover, since $[y_i, C_{p_i}] = M_i$, we have that $[F, X] = \text{Dr}_{i \in I} M_i = [A, X]$. Let $F \leq U \leq G = AK$. Then $AU = AU \cap AK = A(AU \cap K) = AX$, where $X = AU \cap K \leq K$. So, since A is abelian, it follows that:

$$U \geq F^U = F^{AU} = F^{AX} = F^X = F[F, X] = F[A, X].$$

Thus $U \geq [A, X] = [A, AX] = [A, AU] = [A, U] = [AU, U]$, which implies that $U \leq AU \leq G$. Therefore $(G, F) \in \mathfrak{U}_2^+$.

On the other hand, if we look at the lower central series of G , we see that

$$[G, G] = [AK, AK] = [AK, A][AK, K][AK, A, K] = [A, K] = \text{Dr}_{i \in \mathbb{N}} M_i$$

and also that

$$[G, G, G] = [A, K, AK] = [A, K, K] = [\text{Dr}_{i \in \mathbb{N}} M_i, K] = \text{Dr}_{i \in \mathbb{N}} M_i.$$

Therefore $\gamma_2(G) = \gamma_3(G)$ is infinite and we conclude that G is not finite-by-nilpotent. \square

Theorem 2.1. *Let G be a locally nilpotent group and $n \in \mathbb{N}$. If G has a finitely generated subgroup F such that $H \text{ sn}_n G$ whenever $F \leq H \leq G$, then G is hypercentral (and soluble).*

Proof. By assumption, F is nilpotent and subnormal in G . Moreover, by Roseblade's theorem, each section of the normal closure series of F in G is nilpotent. Thus, G is soluble. Clearly, it is sufficient to prove that G has a nontrivial center.

By induction on n , we may assume that $\zeta(F^G) \neq 1$. In particular, F is contained in a normal subgroup N of G which has a nontrivial center. We proceed by induction on the derived length d of G/N .

If $d = 0$, then $G = N$ has a nontrivial center.

Let $d \geq 1$. Set $G_1 = G'N$ and $A = \zeta(G_1)$. Then $A \neq 1$ by induction hypothesis. Also, A is a normal subgroup of G . Let U be a finitely generated subgroup of G containing F . By assumption, U is subnormal of defect at most n in G , so that $[A, {}_n U] \leq U$. Hence, $[A, {}_n U]$ is finitely generated, since U is finitely generated and nilpotent. As $G' \leq C_G(A)$, for every $g \in G$ we get

$$[A, {}_n U]^g = [A, {}_n U^g] \leq [A, {}_n U([U, g])] = [A, {}_n U],$$

so that $[A, {}_n U]$ is normal in G . Therefore, as $[A, {}_n U]$ is finitely generated, $[A, {}_n U] \leq \zeta_k(G)$ for some $k \geq 1$. Thus, if $[A, {}_n U] \neq 1$, then $\zeta(G) \neq 1$.

Otherwise, if $[A, {}_n U] = 1$ for any finitely generated subgroup of G (containing F), then $[A, {}_n G] = 1$, i.e., $A \leq \zeta_n(G)$, and again $\zeta(G) \neq 1$. \square

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